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THE OPTIMAL DIFFERENTIATION BASIS AND LIFTINGS OF L^∞

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ABSTRACT. There is an optimal way to differentiate measures when given a consistent choice of where zero limits must occur. The appropriate differentiation basis is formed following the pattern of an earlier construction by the authors of an optimal approach system for producing boundary limits in potential theory. Applications include the existence of Lebesgue points, approximate continuity, and liftings for the space of bounded measurable functions – all aspects of the fact that for every point outside a set of measure 0, a given integrable function has small variation on a set that is "big" near the point. This fact is illuminated here by the replacement of each measurable set with the collection of points where the set is "big", using a classical base operator. Properties of such operators and of the topologies they generate, e.g., the density and fine topologies, are recalled and extended along the way. Topological considerations are simplified using an extension of base operators from algebras of sets on which they are initially defined to the full power set of the underlying space.

1. INTRODUCTION

Measure differentiation as a limit process takes a suitable measure μ and produces values for its Radon-Nikodým derivative at almost all points of the underlying measure space. As with all limit processes, the existence of a limit at a given point becomes more significant as the "approach" used to obtain that limit becomes coarser. We show here that there is an optimal, i.e., coarsest, way to differentiate measures once one has made a consistent choice of where zero limits must occur. For measures of the form $f\sigma$ produced by bounded functions f and a fixed reference measure σ, the sets where zero limits must occur can be obtained using any lower density operator on measurable sets. In this case, the optimal differentiation basis produces a limit c at a point x if and only if f is essentially close to c at x in terms of the lower density operator.

The construction of our differentiation basis here is analogous to the formation of optimal boundary approach systems for potential theory in [4]. In that setting, points in a domain act as functionals on boundary measures by extending those measures as harmonic functions into the domain and evaluating at the points. As the set of points used at a stage in the limit process gets bigger, that is as the

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approach filter system gets coarser, the information given by the existence of a limit improves. For example, the existence of a nontangential limit at a boundary point gives more information than the existence of a radial limit.

Our method of differentiating measures in a suitable class of measures uses the fact that measurable sets act as functionals on the class: The value of the functional on a measure is the measure of the set. The larger the collection of these functionals one has at any stage of the filtration process, the more information one obtains when a limit exists. We use the principal result from [3] to show that when a reference measure σ and a suitable normalization are given, our differentiation basis is optimal. The normalization assigns to each measure in the class a set of points where the measure derivative must be 0. Known limits, such as those produced by another differentiation basis or some other limit process, often guide in making this assignment. The zero-sets are then used in constructing the filter at each point forming the optimal differentiation basis. The result we use from [3] is a necessary and sufficient condition for almost everywhere convergence; the corresponding filters are the coarsest producing the desired limits. This suggests that other differentiation bases work because they refine the one constructed here.

Underlying various aspects of this theory is the fact that for every point outside a set of measure 0, a given σ-integrable function f has small variation on a set that is "big" near the point. Depending on the context, these points are called "Lebesgue points" and points of "approximate continuity" for f. A measure limit c exists at x with respect to our optimal differentiation basis if and only if x is a Lebesgue point for f. We will also say that f is "essentially close" to c at x. (This is the same as speaking of approximate continuity at x, but ignoring the value at x.) At the end of this article, we will easily construct a lifting for the space of bounded measurable functions $\mathcal{L}^\infty$, which assigns to each member f the value c at any point where f is essential close to c. Along the way, we will recall and further develop the theory of base and antibase operators and of the topologies they generate, such as the density topology and the fine topology from potential theory. Base and antibase operators replace sets with collections of points where the sets are big. With more structure, they are also called upper and lower densities. We will give a new, simple description of the topologies associated with these operators, using a natural extension of the operators from algebras of sets on which they are initially defined to the full power set of the underlying space X. We will even extend base operators to bounded real-valued functions on X, capturing in an elementary way topological operations on the functions.

2. The General Setup

In what follows, $(X, \mathcal{M})$ is a measurable space and M is a convex cone contained in the set of all nonnegative finite measures on $(X, \mathcal{M})$. We use the notation μ_E for the restriction of a measure μ to a measurable set E; i.e., $\mu_E(A) = \mu(A \cap E)$. We assume that for each $\nu \in M$ and $E \in \mathcal{M}$, $\nu_E \in M$. If μ and η are measures in M, we write $\mu \leq \eta$ if $\mu(A) \leq \eta(A)$ for each $A \in \mathcal{M}$. We let $\mathbb{R}^+$ denote the nonnegative real numbers and $\mathbb{N}$ the natural numbers. If X is a metric space, we use the notation $B(x, r)$ for the closed metric ball with center x and radius r.

We work with a fixed nonzero measure $\sigma \in M$, which we call the reference measure. Given $\mu \in M$, we write $d\mu/d\sigma$ for the Radon-Nikodým derivative of the absolutely continuous part of μ with respect to σ. We write $\sigma(A) = 0$ when this equation is true for the σ-completion of $\mathcal{M}$. Note that the class M of measures can consist just of measures absolutely continuous with respect to σ, or more generally, measures obtained from σ using L^p-densities for some p with $1 \le p \le \infty$.

The result we use from [3] is couched in terms of functionals on M. For the purpose of this article, sets $A \in \mathcal{M}$ play the role of functionals. That is, for each $\mu \in M$, the value of the functional determined by A at μ is $\mu(A)$.

3. General Differentiation Bases

In this and some later sections, we work with a measurable subset X_σ of X. For applications, X_σ will be chosen so that all appropriate subsets have positive σ-measure. Our differentiation bases are filter bases on the collection of measurable subsets of X_σ; they are not filter bases on X_σ itself. For example, instead of the family of all balls centered at a point x in a metric space X, we work with sets of balls B_x^δ. Each set B_x^δ consists of all balls centered at x with radius smaller than δ. In general, we use the following variation of the definition in Section 6E of [16].

Definition 3.1. Suppose with each point $x \in X_\sigma$ there is an associated filter base $\mathcal{D}_x$ such that each $D \in \mathcal{D}_x$ is a collection of measurable subsets of X_σ and $\sigma(A) > 0$ for each $A \in D$. A real number c is the **measure limit** of $\mu \in M$ with respect to σ and $\mathcal{D}_x$ at x if, for any $\varepsilon > 0$, there is a set $D \in \mathcal{D}_x$ such that for each $A \in D$ we have $|\mu(A)/\sigma(A) - c| < \varepsilon$. We write $c = \lim_{A \in D \in \mathcal{D}_x} \mu(A)/\sigma(A)$. We will call the mapping $\mathcal{D} : x \mapsto \mathcal{D}_x$ a **differentiation basis** with respect to M and σ on X_σ if for all $\mu \in M$ such a limit exists σ-a.e. on X_σ, and the limits represent the Radon-Nikodým derivative of the absolutely continuous part of μ with respect to σ on X_σ. That is, for all $\mu \in M$,

$$\frac{d\mu}{d\sigma}(x) = \lim_{A \in D \in \mathcal{D}_x} \frac{\mu(A)}{\sigma(A)} \quad \text{for } \sigma\text{-a.e. } x \in X_\sigma.$$

Example 3.2. Fix a metric space (X, ρ), where $\mathcal{M}$ is the class of Borel sets and M is the set of finite, outer regular Borel measures. Assume a result such as the Besicovitch or Vitali covering theorem holds. Let X_σ be the support of the reference measure σ. For each $x \in X_\sigma$ and $\delta > 0$, let B_x^δ be the collection of closed metric balls $B(x, r)$ with $0 < r \le \delta$. The **ball differentiation basis** with respect to σ on X_σ is the mapping $\mathcal{B}$ that associates with each $x \in X_\sigma$ the filter base $\mathcal{B}_x = \left\{B_x^\delta : \delta > 0\right\}$. In [3], Theorem 4.2, a simple proof is given showing that $\mathcal{B}$ is a differentiation basis when the measures in M are inner regular. An obvious modification produces a proof for outer regular measures.

Definition 3.3. We say that a differentiation basis $\mathcal{F}$ is **coarser** than a differentiation basis $\mathcal{D}$ on X_σ if at each $x \in X_\sigma$ the filter generated by $\mathcal{F}_x$ is contained in the filter generated by $\mathcal{D}_x$; we do not require the containment to be strict.

Remark 3.4. Note that the filter generated by $\mathcal{F}_x$ is coarser than the filter generated by $\mathcal{D}_x$ if each set $F \in \mathcal{F}_x$ contains some set $D \in \mathcal{D}_x$. An analogous situation occurs with approaches to the boundary in potential theory for the unit disk: The nontangential approach filter is coarser than the radial approach filter because each nontangential region contains all of the points of some radial segment and more. Note that it is more informative to have a limit with respect to a nontangential region (the coarser filter) than to have it just for a radial approach. Here is a way to make a differentiation basis coarser and then obtain additional information from the existence of the corresponding measure derivatives.

Proposition 3.5. *Let $\mathcal{F} : x \mapsto \mathcal{F}_x$ be a differentiation basis with respect to M and σ on X_σ. Given $x \in X_\sigma$, for each $F \in \mathcal{F}_x$, let*

$$H_F = \left\{ A \in \mathcal{M} : \exists B \in F \text{ with } A \subseteq B \text{ and } \sigma(A) \geq \frac{1}{2}\sigma(B) \right\}.$$

Then $\mathcal{H} : x \mapsto \mathcal{H}_x = \{H_F : F \in \mathcal{F}_x\}$ is a differentiation basis with respect to M and σ on X_σ.

Proof. It is easy to see that for each $x \in X_\sigma$, $\mathcal{H}_x$ is a filter base. Fix a measurable set $E \subseteq X_\sigma$ and a measure $\nu \in M$ with $\nu(E) = 0$. Since $\mathcal{F}$ is a differentiation basis, we know that there is a set $S \subseteq E$ with $\sigma(S) = 0$ such that for each $x \in E \setminus S$, $\lim_{B \in F \in \mathcal{F}_x} \frac{\nu(B)}{\sigma(B)}(x) = 0$. This means that for each $x \in E \setminus S$, we may fix a set $F \in \mathcal{F}_x$ such that $\frac{\nu(B)}{\sigma(B)} \leq \frac{1}{2}$ for each $B \in F$. By definition, for each $A \in H_F$, there is a set $B \in F$ such that $A \subseteq B$ and $\sigma(A) \geq \frac{1}{2}\sigma(B)$. Moreover,

$$\frac{\nu(A)}{\sigma(A)} \leq \frac{\nu(B)}{\sigma(A)} \leq \frac{\nu(B)}{\frac{1}{2}\sigma(B)} = 2 \cdot \frac{\nu(B)}{\sigma(B)} \leq 1.$$

This means that $\limsup_{A \in H_F \in \mathcal{H}_x} \frac{\nu(A)}{\sigma(A)}(x) \leq 1$. By Theorem 1.1 of [3], the result is established. □

Remark 3.6. We will call the differentiation basis $\mathcal{H} : x \mapsto \mathcal{H}_x$ of Proposition 3.5 the **half differentiation basis** formed from $\mathcal{F}$. We could at each point use a positive constant other than 1/2. (See, for example, [9] for the case of balls.) We show next, however, that the value 1/2 plays a special role. That demonstration uses the notation $|\mu|$ for the total variation of a measure μ in M.

Definition 3.7. Let $\mathcal{F} : x \mapsto \mathcal{F}_x$ be a differentiation basis with respect to M and σ on X_σ. A point $x \in X_\sigma$ is a **Lebesgue point** with respect to $\mathcal{F}$, a measure $\mu \in M$, and a constant c if

$$\lim_{B \in F \in \mathcal{F}_x} \frac{|\mu - c \cdot \sigma|(B)}{\sigma(B)} = 0.$$

Remark 3.8. For the case that $\mu = f\sigma$ for some nonnegative σ-integrable function f, the above limit is

$$\lim_{B \in F \in \mathcal{F}_x} \frac{\int_B |f(y) - c|\, \sigma(dy)}{\sigma(B)} = 0.$$

That is, the measure limit is 0 at x for $|f - c|\,\sigma$. For each $\mu \in M$, $|\mu(B) - c \cdot \sigma(B)| \leq |\mu - c \cdot \sigma|(B)$, so a measure derivative exists and equals c at any point that is a Lebesgue point with respect to μ and c. It follows that the constant c is unique for each Lebesgue point. On the other hand, if a measure limit 0 exists for μ at x, then x is a Lebesgue point with respect to μ and 0. Note that a point x is a Lebesgue point with respect to a constant c for a bounded, measurable f if and only if for any $\varepsilon > 0$,

$$\lim_{B \in F \in \mathcal{F}_x} \frac{\sigma(\{x \in B : |f(x) - c| \geq \varepsilon\})}{\sigma(B)} = 0.$$

Remark 3.9. In general, the existence of a measure limit at a point does not mean that the point is a Lebesgue point. For symmetric interval differentiation on the real line, for example, we may let μ be Lebesgue measure multiplied by the characteristic function of $[0, 1]$. The measure derivative with respect to Lebesgue measure exists and is $1/2$ at 0 and 1, but neither point is a Lebesgue point for μ.

Given a differentiation basis, we now isolate the Lebesgue points as the set of points where a measure derivative exists with respect to a coarser differentiation basis constructed from the given one. The authors are indebted to Aurel Cornea for this result (private communication) stated in the context of ball differentiation. It is, as we show, valid in our more general setting.

Theorem 3.10. *Let $\mathcal{F} : x \mapsto \mathcal{F}_x$ be a differentiation basis with respect to M and σ on X_σ. Let $\mathcal{H}$ be the half differentiation basis formed from $\mathcal{F}$. Fix $\mu \in M$, and let $S \subseteq X$ be the set of all points $x \in X_\sigma$ where the measure limit of μ with respect to σ and $\mathcal{H}_x$ exists. Call that limit $f(x)$. Every Lebesgue point x with respect to $\mathcal{F}$, μ and constant c_x belongs to S, and $f(x) = c_x$. Conversely, each $x \in S$ is a Lebesgue point for the differentiation basis $\mathcal{F}$ with respect to the measure μ and the constant $c_x = f(x)$.*

Proof. First assume that x is a Lebesgue point with respect to $\mathcal{F}$, μ and c. Fix $B \in F \in \mathcal{F}_x$, and fix a measurable set $A \subseteq B$ with $\sigma(A) \geq \frac{1}{2}\sigma(B)$. Then

$$\left|\frac{\mu(A) - c \cdot \sigma(A)}{\sigma(A)}\right| \leq \frac{|\mu - c \cdot \sigma|\,(A)}{\sigma(A)} \leq \frac{|\mu - c \cdot \sigma|\,(B)}{\sigma(A)} \leq 2 \cdot \frac{|\mu - c \cdot \sigma|\,(B)}{\sigma(B)}.$$

For the converse, fix $x \in S$ and $\varepsilon > 0$. Set $c = f(x)$. We choose $F \in \mathcal{F}_x$ so that for all $A \in H_F$,

$$\left|\frac{\mu\,(A) - c \cdot \sigma\,(A)}{\sigma\,(A)}\right| < \frac{\varepsilon}{3}.$$

Fix any $B \in F$. Let the sets P and N form a Hahn decomposition of B with respect to $\mu - c \cdot \sigma$. We will assume that $\sigma(P) \geq \frac{1}{2}\sigma\,(B)$; a similar proof works if $\sigma(N) \geq \frac{1}{2}\sigma\,(B)$. Now since B itself is in H_F,

$$|\mu\,(B) - c \cdot \sigma\,(B)| = |\mu\,(P) + \mu(N) - c \cdot \sigma\,(P) - c \cdot \sigma\,(N)| < \frac{\varepsilon}{3} \cdot \sigma\,(B)\,.$$

By assumption,

$$|\mu\,(P) - c \cdot \sigma\,(P)| < \frac{\varepsilon}{3} \cdot \sigma\,(P) \leq \frac{\varepsilon}{3} \cdot \sigma\,(B)\,.$$

It follows that

$$|\mu\,(N) - c \cdot \sigma\,(N)| < \frac{2\varepsilon}{3} \cdot \sigma\,(B)\,,$$

and so

$$|\mu - c \cdot \sigma|\,(B) = \mu\,(P) - c \cdot \sigma\,(P) - \mu\,(N) + c \cdot \sigma\,(N) < \varepsilon \cdot \sigma\,(B)\,.$$

Since ε is arbitrary, we are done. ☐

Corollary 3.11. *Let $\mathcal{F}$ be a differentiation basis with respect to M and σ on X_σ. Fix $\mu \in M$, and let f represent the Radon-Nikodým derivative of the absolutely continuous part of μ with respect to σ. Then σ-almost all points $x \in X_\sigma$ are Lebesgue points with respect to μ and $f(x)$.*

4. Zero-Set Mappings

The process of halving a differentiation basis is an example of producing additional information by coarsening a given differentiation basis. In the next section, we will construct the coarsest, i.e. optimal, differentiation bases that satisfies a consistent specification of where zero limits must occur. The required zeros will be determined by a mapping $Z : \nu \mapsto Z_\nu$ from M into the collection of subsets of X_σ. Each Z_ν will be the set of required zeros for the measure derivative of ν. Here is what we require of the mapping Z.

Definition 4.1. A function Z mapping M into the subsets of X_σ is a **zero-set mapping** if the following five conditions hold for all ν, $\mu \in M$, all measurable sets $E \subseteq X_\sigma$, and all $c > 0$ in $\mathbb{R}$:
i) $Z_\nu \cap Z_\mu \subseteq Z_{\nu+\mu}$,
ii) $Z_{c\nu} = Z_\nu$,
iii) $Z_\nu = \emptyset$ if $\sigma \leq \nu$ on X_σ,
iv) $Z_0 = X_\sigma$,
v) if $\nu(E) = 0$, then $\sigma(E \setminus Z_\nu) = 0$.
Given $\nu \in M$, we call Z_ν the **zero set** for ν. A zero-set mapping is called **positive** if for all ν, $\mu \in M$ we have $Z_\nu \cap Z_\mu = Z_{\nu+\mu}$; that is, $\nu \leq \mu \Rightarrow Z_\mu \subseteq Z_\nu$.

Remark 4.2. The equivalence of the conditions for positivity follows from the facts that, for positive measures, $\nu \leq \nu+\mu$, $\mu \leq \nu+\mu$, and if $\nu \leq \mu$ then $\mu = \nu+(\mu-\nu)$. When τ is a topology on X_σ and $\mathcal{U}_x$ is the collection of τ-open neighborhoods of $x \in X_\sigma$, we will for some results assume the following additional condition holds:

$$\tau)\ \forall x \in Z_\nu,\ \forall U \in \mathcal{U}_x,\ \exists A \in \mathcal{M} \text{ with } A \subseteq U \text{ and } \nu(A) < \sigma(A).$$

Example 4.3. Let X_σ be the support of σ in a metric space where a ball-covering theorem holds. All the above conditions, including Condition τ, will hold if each Z_ν is the set where a limit 0 is obtained when computing a representative for $d\nu/d\sigma$ using the ball differentiation basis on X_σ.

Example 4.4. In the same setting, if M is the space of measures generated by bounded densities on X_σ, we can form a positive zero-set mapping by using a multiplicative (therefore, positive) lifting T of $L^\infty(\sigma)$ with $T(1) \equiv 1$ and $T(0) \equiv 0$ on X_σ. We put $x \in Z_\nu$ if and only if the lifting of the density for ν is zero at x. We can extend this selection to $L^1(\sigma)$ using the following result.

Proposition 4.5. *Let $Z : \nu \mapsto Z_\nu$ be a positive zero-set mapping for measures formed from σ and $L^\infty(\sigma)$ densities. For each nonnegative $h \in L^1(\sigma)$, set $Z_h := Z_{(h\wedge 1)\sigma}$. The mapping $h\sigma \mapsto Z_{h\sigma}$ is a positive zero-set mapping extending the original mapping Z.*

Proof. To show we have an extension of the original mapping Z, we fix $h \in L^\infty(\sigma)$ and a positive constant $\alpha < 1$ so that $\alpha h \leq 1$ σ-a.e. Then

$$Z_{(h\wedge 1)\sigma} \subseteq Z_{(\alpha h\wedge 1)\sigma} = Z_{(\alpha h)\sigma} = Z_{h\sigma} \subseteq Z_{(h\wedge 1)\sigma}.$$

To show that Condition i holds for the extended mapping, we fix two nonnegative functions f and g in $L^1(\sigma)$ and note that

$$(f+g)\wedge 1 \leq (f\wedge 1) + (g\wedge 1)\,.$$

If c is a positive constant, then for $c \le 1$ we have $cf \wedge 1 \le f \wedge 1$, and for $c > 1$ we have

$$cf \wedge 1 = c\left(f \wedge \frac{1}{c}\right) \le c\,(f \wedge 1)\,.$$

It follows from Condition ii that in either case $Z_{(f\wedge 1)\sigma} = Z_{(cf\wedge 1)\sigma}$, whence Condition ii holds for the extended mapping. The rest is clear. □

5. The Optimal Differentiation Basis

Now, in terms of X_σ, $\mathcal{M}$, M, σ, and a given zero-set mapping Z, we construct the coarsest differentiation basis yielding a measure derivative 0 for each $\nu \in M$ at each $x \in Z_\nu$. Given a measure $\nu \in M$, we set

$$D^\nu := \{A \subseteq X_\sigma : A \in \mathcal{M},\, \nu(A) < \sigma(A)\}.$$

For example, $D^0 := \{A \subseteq X_\sigma : A \in \mathcal{M},\, 0 < \sigma(A)\}$. For each $x \in X$ we set

$$\mathcal{D}_x := \{D^\nu : \nu \in M,\ x \in Z_\nu\}.$$

Theorem 5.1. *The mapping $\mathcal{D} : x \mapsto \mathcal{D}_x$ is a differentiation basis on X_σ. It is the coarsest differentiation basis $\mathcal{F}$ with the property that for each $\nu \in M$,*

$$Z_\nu \subseteq \{x \in X : \lim_{A\in F\in\mathcal{F}_x} \frac{\nu(A)}{\sigma(A)} = 0\}.$$

If $y \in X_\sigma$, $\mu \in M$, and $\lim_{A\in D\in\mathcal{D}_y} \frac{\mu(A)}{\sigma(A)} = 0$, then D^μ is in the filter generated by $\mathcal{D}_y$.

Proof. Fix $x \in X_\sigma$. To show that $\mathcal{D}_x$ is a filter base, we note first that by Condition iv for zero sets, $D^0 \in \mathcal{D}_x$, whence $\mathcal{D}_x \ne \emptyset$. Moreover, $\emptyset \notin \mathcal{D}_x$. To see this, fix $\nu \in M$. If $x \in Z_\nu$, then by Condition iii, we must have $\nu(A) < \sigma(A)$ for some measurable subset of X_σ, since otherwise $\sigma \le \nu$, and then $Z_\nu = \emptyset$. It follows that each $D^\nu \in \mathcal{D}_x$ is a nonempty subset of $\mathcal{M}$. If $D^\nu \in \mathcal{D}_x$ and $D^\mu \in \mathcal{D}_x$, then by Condition i for zero sets, $D^{\nu+\mu} \in \mathcal{D}_x$, and of course $D^{\nu+\mu} \subseteq D^\nu \cap D^\mu$. It follows that $\mathcal{D}_x$ is a filter base. To show that the mapping $\mathcal{D}$ is actually a differentiation basis on X_σ, we fix a set $E \in \mathcal{M}$ and a measure $\nu \in M$ with $\nu(E) = 0$. For the σ-completion of $\mathcal{M}$ we have, by Condition v, $\sigma(E \setminus Z_\nu) = 0$, whence $D^\nu \in \mathcal{D}_x$ for σ-a.e. $x \in E$. This means that for σ-a.e. $x \in E$, $\limsup_{A\in D\in\mathcal{D}_x} \frac{\nu(A)}{\sigma(A)} \le 1$. It now follows from Theorem 1.1 of [3] that $\mathcal{D}$ is a differentiation basis on X_σ. To show the required zero limits are obtained, fix $\nu \in M$. If $x \in Z_\nu$, then by Condition ii, for each $n \in \mathbb{N}$, $D^{n\nu} \in \mathcal{D}_x$, and for each $A \in D^{n\nu}$, $n\nu(A) < \sigma(A)$, whence $\nu(A)/\sigma(A) < 1/n$. Therefore, $\lim_{A\in D\in\mathcal{D}_x} \frac{\nu(A)}{\sigma(A)} = 0$. If $\mu \in M$ and $\mathcal{F}$ is a differentiation basis on X_σ for which $\lim_{A\in F\in\mathcal{F}_x} \frac{\mu(A)}{\sigma(A)} = 0$, then there must be a set $F \in \mathcal{F}_x$ such that for each $B \in F$, $\mu(B) < \sigma(B)$. This means that D^μ must be an element of the filter generated by $\mathcal{F}_x$. In particular, if $\lim_{A\in D\in\mathcal{D}_x} \frac{\mu(A)}{\sigma(A)} = 0$, D^μ is in the filter generated by $\mathcal{D}_x$. It now follows that the mapping $x \mapsto \mathcal{D}_x$ is the coarsest differentiation basis on X_σ yielding the zeros specified by the mapping Z. □

Remark 5.2. The last statement of Theorem 5.1 shows that producing the differentiation basis $x \mapsto \mathcal{D}_x$ is an idempotent operation. Once the sets of zeros Z_ν, $\nu \in M$, are produced by using the differentiation basis $\mathcal{D}$, they do not change if the process is repeated. Moreover, if the zero-set mapping Z is formed to begin with by the zeros resulting from using a differentiation basis $\mathcal{F}$, then since the resulting differentiation basis $\mathcal{D}$ is coarser than $\mathcal{F}$, using $\mathcal{D}$ produces no new zeros.

Corollary 5.3. *The differentiation basis $\mathcal{D}$ is its own half differentiation basis. Therefore a measure limit for $\mu \in M$ exists at x if and only if x is a Lebesgue point with respect to $\mathcal{D}$ and μ.*

Proof. Let $\mathcal{H}$ be the half differentiation basis constructed from $\mathcal{D}$. Given $\nu \in M$, if $x \in Z_\nu$ then $\mathcal{D}_x$ produces a zero limit for ν at x, so x is a Lebesgue point with respect to $\mathcal{D}$ and ν. It follows from Theorem 3.10 that $\mathcal{H}_x$ also produces a zero limit for ν at x. But $\mathcal{D}$ is the coarsest differentiation basis producing the zeros specified by Z. Therefore, $\mathcal{H} = \mathcal{D}$. □

For the rest of this section, we suppose there is a topology τ on X_σ. Let $\mathcal{U}_x$ denote the collection of τ-open neighborhoods of $x \in X_\sigma$. Unlike the ball differentiation basis and its corresponding half differentiation bases, the optimal differentiation basis $\mathcal{D}$ of Theorem 5.1 does not converge to points in X_σ in the sense that for any $U \in \mathcal{U}_x$ there is a $D^\nu \in \mathcal{D}_x$ such that $A \subseteq U$ for all $A \in D^\nu$. Recall Condition τ:

$$\forall x \in Z_\nu,\ \forall U \in \mathcal{U}_x,\ \exists A \in \mathcal{M} \text{ with } A \subseteq U \text{ and } \nu(A) < \sigma(A).$$

If this additional condition holds for zero sets (as is the case, for example, when ball differentiation defines the map Z), then we can remedy the situation as follows.

Theorem 5.4. *Assume the zero-set mapping Z satisfies Condition τ. For each $\nu \in M$ and each $U \in \tau$, set $D_U^\nu := \{A \in D_\nu : A \subseteq U\}$. Given $x \in X_\sigma$, set*

$$\mathcal{D}_x^\tau := \{D_U^\nu : \nu \in M,\ x \in Z_\nu,\ U \in \mathcal{U}_x\}.$$

The mapping $\mathcal{D}^\tau : x \mapsto \mathcal{D}_x^\tau$ is a differentiation basis on X_σ. It is the coarsest differentiation basis refining $\mathcal{D}$ such that for each $x \in X_\sigma$ and each $U \in \mathcal{U}_x$ there is an element D of $\mathcal{D}_x^\tau$ such that $A \subseteq U$ for all $A \in D$.

Proof. Fix $x \in X_\sigma$. By Condition τ, $\emptyset \notin \mathcal{D}_x^\tau$. It is thus easy to see that $\mathcal{D}_x^\tau$ is a filter base that refines $\mathcal{D}_x$. Since the limits obtained using $\mathcal{D}$ are retained using a refinement, $\mathcal{D}^\tau$ is a differentiation basis. □

6. Base Operators and Topologies

Applications and extensions of the previous results are conveniently stated in terms of base operators and the corresponding topologies. In this and the next section, we recall and expand on some of the relevant literature, e.g., [10], [14], and [16]. We work with a set X, which in our applications will be X_σ. We use the notation 1_A for the characteristic function, and $\complement A$ for the complement, in X, of a subset A. We write $\mathcal{P}(X)$ for the full power set of X. The set where a property p holds in X may be denoted by $\{p\}$.

Rather than a full measure structure, we work in this section with an algebra $\mathcal{A}$ of subsets of X; we may have $\mathcal{A} = \mathcal{P}(X)$. Given set mappings s and $t : \mathcal{A} \to \mathcal{P}(X)$, we write $s \preceq t$ if for each $A \in \mathcal{A}$, $s(A) \subseteq t(A)$. We will use $\mathcal{F}_b$ to denote the space of bounded real-valued functions on X.

Definition 6.1. Recall that a **base** operator, or just a base, on an algebra $\mathcal{A}$ of subsets of X is a mapping $b : \mathcal{A} \to \mathcal{P}(X)$ such that

$$b(\emptyset) = \emptyset,\ \ b(X) = X,\ \text{ and }\ \forall A, B \in \mathcal{A},\ b(A \cup B) = b(A) \cup b(B).$$

We will call e an **antibase** operator, or just an antibase, on $\mathcal{A}$ if $e : \mathcal{A} \to \mathcal{P}(X)$ has the property that

$$e(\emptyset) = \emptyset, \; e(X) = X, \; \text{and} \; \forall A, B \in \mathcal{A}, \; e(A \cap B) = e(A) \cap e(B).$$

The antibase e constructed from a base b by setting $e(A) = \complement(b(\complement A))$ is called the **antibase associated with** b. We have a similar definition for the **base associated with an antibase**.

Remark 6.2. It is easy to prove that every base and antibase operator is an increasing set mapping. If b and e are associated base and antibase operators, then $e \preceq b$, since for each $A \in \mathcal{A}$, $b(A) \cup b(\complement A) = X$, so $e(A) = \complement b(\complement A) \subseteq b(A)$. Note that a base that is also an antibase equals its associated antibase.

We now fix a nonstandard enlargement of a structure containing X and the real numbers (see, for example, [13]). We write $\alpha \approx \beta$ when α and β are infinitely close nonstandard real numbers. We need the fact that every limited (i.e., finite) nonstandard number α is infinitely close to a unique real number a; we write $a = \operatorname{st} \alpha$. For the nonstandard extensions of sets A and B, we have ${}^*(A \cup B) = {}^*A \cup {}^*B$, ${}^*(A \cap B) = {}^*A \cap {}^*B$, and ${}^*\complement A = \complement {}^*A$. Recall that the **monad** of a point x in a topological space is the set $\bigcap {}^*O$, where the intersection is taken over the collection of all standard open sets O that contain x. We will also use the fact that if $\mathcal{F}$ is a filter base in X, then there is an (internal) set $B \in {}^*\mathcal{F}$ with $B \subseteq \bigcap_{A \in \mathcal{F}} {}^*A$.

Definition 6.3. We call a mapping $E : X \to \mathcal{P}({}^*X) \setminus \{\emptyset\}$ a **base generating function**. We write E_x for the nonempty (possibly external) set that is the image of x. The operators b and e on $\mathcal{P}(X)$ given by

$$b(A) := \{x \in X : E_x \cap {}^*A \neq \emptyset\} \quad \text{and} \quad e(A) := \{x \in X : E_x \subseteq {}^*A\}$$

are called the **base and antibase on $\mathcal{P}(X)$ generated by** E.

Example 6.4. A simple example of a base generating function is obtained for a topological space X by letting E_x be the monad of x for each $x \in X$. In this case, $e(A)$ is the interior of A and $b(A)$ is the closure of A for each $A \subseteq X$.

Remark 6.5. Given a base generating function, it is easy to check that b and e are in fact a base and antibase associated with each other on the algebra $\mathcal{P}(X)$. As we now show, every base and associated antibase on an algebra $\mathcal{A}$ can be obtained from a base generating function. This gives an automatic extension of these operators to the full power set of X.

Theorem 6.6. *Fix an algebra $\mathcal{A}$ of subsets of X together with a base operator b and the corresponding antibase operator e on $\mathcal{A}$. For each $x \in X$, we set*

$$E_x := \bigcap \{{}^*A : A \in \mathcal{A}, \, x \in e(A)\}.$$

The mapping E is a base generating function on X, and the corresponding base and antibase generated by E on the full power set of X are extensions of b and e.

Proof. If $x \in e(A)$ and $e(B)$, then $x \in e(A \cap B)$, so $A \cap B \neq \emptyset$. It follows that $E_x \neq \emptyset$. Fix $A \in \mathcal{A}$ and $x \in X$. If $x \in e(A)$, then by definition, $E_x \subseteq {}^*A$. If $E_x \subseteq {}^*A$, then there is an internal set $B \in {}^*\mathcal{A}$ with $B \subseteq E_x$ and $x \in {}^*e(B) \subseteq {}^*e(A)$, whence $x \in e(A)$. It follows that for each $A \in \mathcal{A}$, $e(A) = \{x \in X : E_x \subseteq {}^*A\}$. Moreover, for each $A \in \mathcal{A}$,

$$x \in b(A) \Leftrightarrow x \notin e(\complement A) \Leftrightarrow E_x \cap {}^*A \neq \emptyset. \quad \square$$

Remark 6.7. If the antibase e is originally defined on $\mathcal{A}$ using a base generating function F, then for each $x \in X$, $F_x \subseteq E_x$. The theorem shows that both F and E generate the same base b and antibase e on $\mathcal{A}$. We will call E the **base generating function associated with** b **and** e on $\mathcal{A}$.

Example 6.8. An important example of an antibase is the mapping e on a suitable metric space that sends each measurable set A to its points of density with respect to ball differentiation. In this case, for each $x \in X$, the set E_x is the intersection of all sets ${}^*A$ such that x is a point of density of A. Of course, this is a subset of the monad of x.

Definition 6.9. Let E be a base generating function on X. For each $f \in \mathcal{F}_b$ we set

$$b_E(f)(x) := \sup \{\mathrm{st}\,({}^*f(z)) : z \in E_x\},$$
$$e_E(f)(x) := \inf \{\mathrm{st}\,({}^*f(z)) : z \in E_x\} = -b_E(-f)(x).$$

We call a mapping $\phi : X \to {}^*X$ with $\phi(x) \in E_x$ for each $x \in X$ a **base choice function** with respect to E. Given such a function, for each $A \in \mathcal{P}(X)$ and each $f \in \mathcal{F}_B$, we set

$$t_\phi(A) := \{x \in X : \phi(x) \in {}^*A\}, \text{ and } t_\phi(f)(x) := \mathrm{st}\,({}^*f(\phi(x))).$$

Remark 6.10. It is easy to see that for each set $A \subseteq X$, $1_{b(A)} = b_E(1_A)$ and $1_{e(A)} = e_E(1_A)$. In selecting a base choice function, if $x \in E_x$, we may choose $\phi(x) = x$. On the other hand, given a set A_0 we may choose $\phi(x) \in {}^*A_0 \cap E_x$ if this set is nonempty; we then have $t_\phi(A_0) = b(A_0)$. Similarly, we may choose $\phi(x) \in {}^*\complement A_0 \cap E_x$ if this set is nonempty; we then have $t_\phi(A_0) = e(A_0)$.

Proposition 6.11. *Let E be a base generating function and ϕ a base choice function with respect to E. Let b and e be the base and antibase operators on $\mathcal{P}(X)$ generated by E. Then*

1. $e \preceq t_\phi \preceq b$, *and for each* $f \in \mathcal{F}_b$, $e_E(f) \le t_\phi(f) \le b_E(f)$.
2. *The operator t_ϕ is both a base and an antibase on $\mathcal{P}(X)$.*
3. *For any pair of functions f and g in $\mathcal{F}_b$ and any constant α,* $t_\phi(f+g) = t_\phi(f) + t_\phi(g)$, $t_\phi(f \cdot g) = t_\phi(f) \cdot t_\phi(g)$, $t_\phi(\alpha f) = \alpha \cdot t_\phi(f)$.
4. $t_\phi(f)(x) = a$ *if* ${}^*f(y) \approx a$ *for all* $y \in E_x$.

Proof. Clear. □

Proposition 6.12. *Let b be a base operator on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, and let E be the associated base generating function. If f is a nonnegative function in $\mathcal{F}_b$, then for all $x \in X$,*

$$b_E(f)(x) = \sup \{t \ge 0 : x \in b(\{f \ge t\})\}.$$

It follows that if $\mathcal{A}$ is actually a σ-algebra and $b : \mathcal{A} \to \mathcal{A}$, then $b_E(f)$ is $\mathcal{A}$-measurable when $f \ge 0$ is $\mathcal{A}$-measurable.

Proof. The result follows from the fact that

$$\begin{aligned} b_E(f)(x) &= \sup \{t \ge 0 : E_x \cap \{{}^*f \ge t\} \ne \emptyset\} \\ &= \sup \{t \ge 0 : x \in b(\{f \ge t\})\}. \end{aligned}$$

□

We now fix an algebra $\mathcal{A}$ of subsets of X together with a base operator b and the associated antibase operator e on $\mathcal{A}$. We let E be the associated base generating function. We also let b and e denote the extensions of b and e to the full power set of X. The **b-topology**, described in [16], is given by the following characterization: A set $O \subseteq X$ is open in the b-topology iff $O \subseteq e(O)$. Equivalently, a set $F \subseteq X$ is closed in the b-topology iff $b(F) \subseteq F$. It is easy to see that this definition does in fact produce a topology on X.

Proposition 6.13. *A set $O \subseteq X$ is open in the b-topology iff for each $x \in O$, $E_x \subseteq {}^*O$.*

Proof. Assume O is open in the b-topology. If $x \in O$, then $x \in e(O)$, so $E_x \subseteq {}^*O$. Conversely, assume that for each $y \in O$, $E_y \subseteq {}^*O$. Then for any given $x \in O$, there is an internal set $B \in {}^*\mathcal{A}$ with $B \subseteq E_x \subseteq {}^*O$ and $x \in {}^*e(B) \subseteq {}^*e(O)$. It follows that $x \in e(O)$. This shows that $O \subseteq e(O)$, i.e., O is open in the b-topology. □

A base operator b is called **strong** on $\mathcal{A}$ if for each $A \in \mathcal{A}$, $b(b(A)) \subseteq b(A)$. An equivalent condition for the associated antibase is that for each $A \in \mathcal{A}$ we have $e(e(A)) \supseteq e(A)$. Recall that a topological space is T_1 if each point x forms a closed singleton set. That is, for any point $y \neq x$ in X, the monad of y does not contain x. For the b-topology, this means $b(\{x\}) \subseteq \{x\}$ for each $x \in X$.

Theorem 6.14. *Assume that b is a strong base operator on $\mathcal{A}$. Then for each $x \in X$, the monad of x in the b-topology is $E_x \cup \{x\}$. Conversely, b is strong if the b-topology is T_1 and $E_x \cup \{x\}$ is the monad of x.*

Proof. Fix $x \in X$. By Proposition 6.13,

$$E_x \cup \{x\} = \bigcap \{{}^*A \cup \{x\} : A \in \mathcal{A},\, x \in e(A)\} \subseteq \bigcap \{{}^*O : O \text{ open},\, x \in O\}.$$

We must show that if b is strong, the above set containment can be replaced with equality. Given $A \in \mathcal{A}$ with $x \in e\,(A)$, the set $A \cap e(A)$ is an open subset of A since

$$e\,(A \cap e(A)) = e(A) \cap e(e(A)) = e(A) \supseteq A \cap e(A).$$

Moreover, as we have shown, $e\,(A \cap e(A)) = e(A)$, so $x \in e\,(A \cap e(A))$. It follows that $E_x \subseteq {}^*\,(A \cap e(A))$, whence $(A \cap e(A)) \cup \{x\}$ is an open neighborhood of x.

To establish the converse, we fix an $A \in \mathcal{A}$ and show that $b(A)$ is closed. It follows from our assumption that $b(A)$ is the set of all accumulation points of A together with those points $x \in A$ that are also in E_x. Any accumulation point of the set of accumulation points of A is an accumulation point of A. Therefore, $b(A)$ is closed. □

Example 6.15. Fix a two point set $\{\alpha, \beta\}$. Let $E_\alpha = \{\beta\}$ and $E_\beta = \{\alpha\}$. Then $e\,(\{\alpha\}) = b\,(\{\alpha\}) = \{\beta\}$, and $e\,(\{\beta\}) = b\,(\{\beta\}) = \{\alpha\}$. The b-topology is the trivial topology, so the monads of α and β are each equal to $\{\alpha, \beta\}$. Of course, $b(b(\{\alpha\})) = b\,(\{\beta\}) = \{\alpha\}$, which is not a subset of $b(\{\alpha\}) = \{\beta\}$.

Definition 6.16. Given a strong base operator b and associated antibase e, we say that a real-valued function f defined on X is **essentially close** to a constant c at x, and that c is the **approximate value** for f at x, if for any $\varepsilon > 0$, $x \in e\,(\{y \in X : |f(y) - c| < \varepsilon\})$. In standard terminology, a function is said to be **approximately continuous at a point** if it is continuous there in the b-topology.

Proposition 6.17. *Fix a strong base operator b and associated antibase e. Let f be a real-valued function defined on X.*

1. *There is at most one approximate value for f at any point of X.*
2. *c is an approximate value for f at x iff for each $y \in E_x$, ${}^*f(y) \approx c$.*
3. *f is continuous at x in the b-topology iff $f(x)$ is an approximate value for f at x.*

Proof. The uniqueness of approximating values follows from the fact that disjoint sets have disjoint images under e. It follows from the definition that if c is an approximate value of f at x, then ${}^*f$ differs from c by at most an infinitesimal on E_x. For the converse, we use the fact that there is an internal set $B \in {}^*\mathcal{M}$ with $B \subseteq E_x$ and $x \in {}^*e(B)$. If ${}^*f(y) \approx c$ for all $y \in B$, then for any $\varepsilon > 0$ in $\mathbb{R}$, there is by "downward transfer" a standard set $A \in \mathcal{M}$ with $x \in e(A)$ such that $|f(y) - c| < \varepsilon$ for all $y \in A$. It follows then that c is an approximate value for f at x. The last statement follows from Theorem 6.14: Since b is a strong base operator, the monad of any $x \in X$ is the set $E_x \cup \{x\}$. □

Remark 6.18. An important base operator $b : \mathcal{P}(X) \to \mathcal{P}(X)$ occurs in potential theory when X is the underlying set of a balayage space or a harmonic space (see [1]). For any set $A \subseteq X$, the base $b(A)$ is defined by setting

$$b(A) := \{x \in X : A \text{ is not thin at } x\}.$$

The base operator b is strong, and the b-topology is just the fine topology. By Theorem 6.14, the fine topology monad of each $x \in X$ is the set $E_x \cup \{x\}$. We should note that the standard formula for b_E in Proposition 6.12 was used by Fuglede [10] in a potential theoretic context to extend the base operator to functions. Another important example of a strong base operator is given by the essential base β associated with b. (See [1] or [16].) This base β has the properties that $\beta^2 = \beta$, $\beta \preceq b$, and $\beta = b$ iff $\beta(A) = \emptyset \Rightarrow b(A) = \emptyset$. We note that the base generating function for b is given at $x \in X$ by

$$E_x := \bigcap \{{}^*V : V \text{ a fine neighborhood of } x\}$$

if $b(\{x\}) = x$, but one must remove x if $b(\{x\}) = \emptyset$. On the other hand, the base generating function for β is given at x by

$$E_x := \bigcap \{{}^*(V \setminus S) : V \text{ a fine neighborhood of } x,\ \ S \text{ a semipolar set}\}.$$

7. Densities and Topologies

We now fix an algebra $\mathcal{A}$ in $\mathcal{P}(X)$ and a collection $\mathcal{N} \subseteq \mathcal{A}$ stable under the operation of taking finite unions, and also having the property that any subset of an element of $\mathcal{N}$ is again in $\mathcal{N}$, and therefore also in $\mathcal{A}$. We call $\mathcal{N}$ a **collection of null sets** in $\mathcal{A}$. For arbitrary subsets A and B of X, we write $A \sim B$ if the symmetric difference $(A \setminus B) \cup (B \setminus A)$ is in $\mathcal{N}$.

Definition 7.1. A base operator b is called an **upper density** (with respect to $\mathcal{N}$) if $b(A) \setminus A \in \mathcal{N}$ for each $A \in \mathcal{A}$ and $b(A) = \emptyset$ if $A \in \mathcal{N}$. An antibase operator e is called a **lower density** (with respect to $\mathcal{N}$) if $A \setminus e(A) \in \mathcal{N}$ for each $A \in \mathcal{A}$ and $e(A) = X$ if $A \sim X$. An upper density that is also a lower density is called a **lifting** of $\mathcal{A}$.

Proposition 7.2. *If b is an upper density on $\mathcal{A}$ with respect to a collection of null sets $\mathcal{N}$, then the associated antibase e is a lower density. A similar fact is true if e is a lower density with respect to $\mathcal{N}$. In either case, for each set $A \in \mathcal{A}$, we* *have* $e(A) \sim A$ *and* $b(A) \sim A$. *Moreover, if* $A \sim B$ *in* $\mathcal{A}$, *then* $e(A) = e(B)$ *and* $b(A) = b(B)$.

Proof. Suppose b is an upper density and e the associated antibase. Given $A \in \mathcal{A}$, we have $b(A) \cap \complement A \in \mathcal{N}$ and $b(\complement A) \cap A \in \mathcal{N}$. Since $e \preceq b$, $e(\complement A) \cap A \in \mathcal{N}$, i.e., $A \setminus b(A) \in \mathcal{N}$. It follows that $b(A) \sim A$. If $A \sim B$ in $\mathcal{A}$, then $b(A) = b(A \cap B) \cup b(A \setminus B) = b(A \cap B) = b(B)$. It now follows that for each set $A \in \mathcal{A}$, $e(A) = \complement b(\complement A) \sim \complement\complement A = A$, and if $A \sim X$, $e(A) = \complement b(\complement A) = \complement b(\emptyset) = X$. Suppose that e is a lower density with respect to $\mathcal{N}$ and b the associated base. Given $A \in \mathcal{A}$, we have $A \cap \complement e(A) \in \mathcal{N}$ and $\complement A \cap b(A) \in \mathcal{N}$, whence $\complement A \cap e(A) \in \mathcal{N}$, i.e., $e(A) \setminus A \in \mathcal{N}$. It follows that $e(A) \sim A$. For each $A \in \mathcal{A}$, $b(A) = \complement e(\complement A) \sim \complement\complement A = A$, and if $A \in \mathcal{N}$, then $b(A) = \complement e(\complement A) = \complement X = \emptyset$. Therefore, b is an upper density. If $A \sim B$ in $\mathcal{A}$, then $e(A) = \complement b(\complement A) = \complement b(\complement B) = e(B)$. □

Proposition 7.3. *Suppose* b *and* e *are associated upper and lower densities with respect to* $\mathcal{N}$ *on* $\mathcal{A}$ *and* E *is the associated base generating function. Then the following properties hold for* E:

$$\text{i) } \forall A \in \mathcal{A},\ \left\{x \in A : E_x \cap \complement^* A \neq \emptyset\right\} \in \mathcal{N},$$
$$\text{ii) } \forall A \in \mathcal{N},\ \forall x \in X,\ \ E_x \subseteq \complement^* A.$$

Conversely, suppose that E *is any base generating function satisfying Properties i and ii. Then the base* b *and antibase* e *generated by* E *are upper and lower densities with respect to* $\mathcal{N}$ *on* $\mathcal{A}$.

Proof. Both statements follow from the fact that for each $A \in \mathcal{A}$,

$$b(A) \setminus A = \left\{x \in \complement A : E_x \cap {}^* A \neq \emptyset\right\},$$
$$A \setminus e(A) = \left\{x \in A : E_x \cap {}^* \complement A \neq \emptyset\right\},$$

and $b(A) = \emptyset$ iff $E_x \cap {}^*A = \emptyset$ for all $x \in X$, while $e(A) = X$ iff $E_x \subseteq {}^*A$ for all $x \in X$. □

Proposition 7.4. *Fix an upper density* b *and its associated lower density* e *with respect to* $\mathcal{N}$ *on the algebra* $\mathcal{A}$. *For each* $A \in \mathcal{A}$, $b(b(A)) = b(A)$, *so* b *is a strong base operator. Let* E *be the base generating function associated with* b *and* e. *For any* ϕ *that is a base choice function with respect to* E, *we have* $t_\phi(A) \sim A$ *for each* $A \in \mathcal{A}$. *If, moreover,* e *and* b *are equal, i.e.,* e *is a lifting of* $\mathcal{A}$, *then for any base choice function* ϕ *with respect to* E *we have* $e = t_\phi$ *on* $\mathcal{A}$.

Proof. If ϕ is a base choice function, then for each $A \in \mathcal{A}$, $e(A) \subseteq t_\phi(A) \subseteq b(A)$, so $t_\phi(A) \sim A$. If, moreover, $e = b$, then $e = t_\phi$ on $\mathcal{A}$. □

Remark 7.5. When b is an upper density, the b-topology is called an **abstract density topology** (see Section 6E of [16]). If $\{x\} \in \mathcal{N}$, then $x \in e\left(X \setminus \{x\}\right) = X$, so $E_x \subseteq {}^*\left(X \setminus \{x\}\right)$, whence $x \notin E_x$. If $\{x\} \notin \mathcal{N}$, then $e\left(\{x\}\right) \sim \{x\}$, so $x \in e\left(\{x\}\right)$. In this case, $E_x \subseteq {}^*\{x\}$, which means that $E_x = \{x\}$, and $\{x\}$ is both open and closed in the b-topology.

Proposition 7.6. *If* $\{x\} \in \mathcal{N}$ *and* $f \in \mathcal{F}_b$, *the values* $b_E(f)(x)$ *and* $e_E(f)(x)$ *are the* lim sup *and* lim inf *of* f *at* x *in terms of the deleted* b*-neighborhood system of* x. *That is,*

$$b_E(f)(x) = \limsup_{y \to x,\ y \neq x} f(y), \quad \text{and} \quad e_E(f)(x) = \liminf_{y \to x,\, y \neq x} f(y).$$

Proof. Clear. □

Remark 7.7. Our observations on approximate continuity and the density monad generalize results in Proposition IV.4 and its preceding remark that were established by Wattenberg for $[0,1]$ in [18]. Luxemburg has also used nonstandard analysis in arguments involving points of density on the real line in [17].

8. $\mathcal{L}^\infty$ and Approximate Values

Returning to our measure theoretic setup, we now restrict our attention to measures that are absolutely continuous with respect to the finite measure σ. As we have seen, we may even restrict our attention to measures produced by nonnegative bounded densities. From now on, we write X instead of X_σ, and we let $\mathcal{M}$ be the algebra of measurable subsets of X and $\mathcal{N}$ the σ-null sets. We assume that $(X, \mathcal{M}, \sigma)$ is a complete measure space. We use $\mathcal{L}^\infty$ to denote the space of bounded measurable functions on X.

Proposition 8.1. *Let Z be a positive zero-set mapping for the measures of the form $f\sigma$ where $f \geq 0$ is in $\mathcal{L}^\infty$. For each $A \in \mathcal{M}$, let $e(A) := Z_{1_{\complement A}\sigma} = Z_{\sigma_{\complement A}}$. Then e is a lower density on $\mathcal{M}$. Moreover, if $x \in Z_{f\sigma}$, then for each $n \in \mathbb{N}$, $x \in e(\{f < 1/n\})$.*

Proof. By definition, $e(\emptyset) = Z_\sigma = \emptyset$. Given sets A and B in $\mathcal{M}$, if $A \sim X$, then $e(A) = Z_{1_{\complement A}\sigma} = Z_0 = X$. If $x \in e(A) \cap e(B) = Z_{1_{\complement A}\sigma} \cap Z_{1_{\complement B}\sigma} = Z_{(1_{\complement A}+1_{\complement B})\sigma}$, then $x \in Z_{1_{(\complement A)\cup(\complement B)}\sigma} = Z_{1_{\complement(A\cap B)}\sigma}$, since $1_{(\complement A)\cup(\complement B)} \leq 1_{\complement A} + 1_{\complement B}$. It follows that $e(A) \cap e(B) \subseteq e(A \cap B)$. The reverse containment follows from the fact that if $x \in e(A \cap B)$, then $x \in Z_{1_{(\complement A)\cup(\complement B)}\sigma} \subseteq Z_{1_{\complement A}\sigma}$, so $x \in e(A)$. Similarly, $x \in e(B)$, so $x \in e(A) \cap e(B)$. Since $\sigma_{\complement A}(A) = 0$, we have $\sigma(A \setminus e(A)) = \sigma\left(A \setminus Z_{\sigma_{\complement A}}\right) = 0$, so e is a lower density. Finally, fix $n \in \mathbb{N}$ and $x \in Z_{f\sigma}$; let $B = \{f < 1/n\}$. Then $x \in Z_{f\cdot 1_{\complement B}\sigma}$, so $x \in Z_{n\cdot f\cdot 1_{\complement B}\sigma} \subseteq Z_{1_{\complement B}\sigma}$, whence $x \in e(B)$. □

Example 8.2. Suppose Z is the zero-set mapping produced by measure differentiation with respect to balls and Lebesgue measure on a bounded open subset of $\mathbb{R}^n$. Then $e(A)$ is exactly the set of points of density of A.

Rather than starting with a zero mapping for measures generated by $\mathcal{L}^\infty$ and σ, we will for the rest of this section assume that we have a fixed lower density e and associated upper density b mapping $\mathcal{M}$ into $\mathcal{M}$. The null sets $\mathcal{N}$ are the sets of σ-measure 0. We will use e to form a zero-set mapping for the measures generated by $\mathcal{L}^\infty$. As before, for each $x \in X$ we set $E_x := \bigcap \{{}^*A : A \in \mathcal{M},\, x \in e(A)\}$.

For each nonnegative $f \in \mathcal{L}^\infty$, we let Z_f be the set of point where f is essentially close to 0. That is,

$$Z_f := \bigcap_{n=1}^{\infty} e\left(\left\{f < \frac{1}{n}\right\}\right) = \{x \in X : b_E(f)(x) = 0\}.$$

Notice we use the notation Z_f instead of $Z_{f\sigma}$ for this set mapping.

Proposition 8.3. *The mapping $f\sigma \mapsto Z_f$ is a positive zero-set mapping on the measures generated by $\mathcal{L}^\infty$ and σ.*

Proof. Fix nonnegative functions f and g in $\mathcal{L}^\infty$. If $f = g$ σ-a.e., then $Z_f = Z_g$. If $f \leq g$, then for each $n \in \mathbb{N}$, $\{g < 1/n\} \subseteq \{f < 1/n\}$, so $Z_g \subseteq Z_f$. Since $b_E(f)(x) = 0$ and $b_E(g)(x) = 0$ if and only if $b_E(f+g)(x) = 0$, it follows that $Z_f \cap Z_g = Z_{f+g}$. If

$x \in Z_f$, then for each positive number α, we have $b_E(\alpha f)(x) = 0$, whence $x \in Z_{\alpha f}$. If $f \geq 1$ σ-a.e., then by Proposition 7.3, $Z_f = \emptyset$. Moreover, $Z_0 = e(X) = X$. Finally, assume $f = 0$ σ-a.e. on a set $A \in \mathcal{M}$. Let $B := \{x \in A : f(x) > 0\}$. For σ-a.e. $x \in A$ we have $x \in e(A)$, so by Proposition 7.3, $E_x \subseteq {}^*A \setminus {}^*B$, whence $b_E(f)(x) = 0$. That is, $\sigma(A \setminus Z_f) = 0$. □

We now want to relate measure differentiation and approximate values for functions in $\mathcal{L}^\infty$. Because we can always add a positive constant to any $f \in \mathcal{L}^\infty$, we may restrict our attention to nonnegative functions.

Theorem 8.4. *Let $\mathcal{D}$ be the optimal differentiation basis formed using the construction of Section 5 from the mapping $f\sigma \mapsto Z_f$ defined above. A nonnegative function f in $\mathcal{L}^\infty$ is essentially close to c at x with respect to the lower density e iff*

$$\lim_{A \in D \in \mathcal{D}_x} \frac{f\sigma(A)}{\sigma(A)} = c.$$

In particular,

$$Z_f = \left\{ x \in X : \lim_{A \in D \in \mathcal{D}_x} \frac{f\sigma(A)}{\sigma(A)} = 0 \right\}.$$

Proof. By Corollary 5.3, $\mathcal{D}$ is its own half differentiation basis, so

$$\lim_{A \in D \in \mathcal{D}_x} \frac{f\sigma(A)}{\sigma(A)} = c \quad \text{iff} \quad \lim_{A \in D \in \mathcal{D}_x} \frac{|f - c|\,\sigma(A)}{\sigma(A)} = 0.$$

We may, therefore, restrict the proof to the case of the constant $c = 0$. Let $M = \|f\|_\infty$. Assume first that f is essentially close to 0 at x. Fix $n \in \mathbb{N}$, and let $S = \{f < 1/n\}$. Then $x \in e(S)$. Let $g_n := n \cdot 1_{\complement S}$, and let $\nu_n = g_n \sigma$. For each ε with $0 < \varepsilon < 1$ we have $\{g_n < \varepsilon\} = S$, so $x \in Z_{g_n}$. By definition, one element of the filter base $\mathcal{D}_x$ is the collection

$$\begin{aligned} D^{\nu_n} &= \{A \in \mathcal{M} : \nu_n(A) < \sigma(A)\} = \{A \in \mathcal{M} : n\sigma(A \setminus S) < \sigma(A)\} \\ &= \left\{ A \in \mathcal{M} : \frac{\sigma(A \setminus S)}{\sigma(A)} < \frac{1}{n} \right\}. \end{aligned}$$

For each $A \in D^{\nu_n}$,

$$\frac{\int_A f\, d\sigma}{\sigma(A)} \leq \frac{1}{n} \cdot \frac{\sigma(A \cap S)}{\sigma(A)} + \frac{M \cdot \sigma(A \setminus S)}{\sigma(A)} \leq \frac{1}{n} + \frac{M}{n}.$$

This shows that

$$\lim_{A \in D \in \mathcal{D}_x} \frac{f\sigma(A)}{\sigma(A)} = 0.$$

Now assume that $\lim_{A \in D \in \mathcal{D}_x} \frac{f\sigma(A)}{\sigma(A)} = 0$. Given $\varepsilon > 0$, there is a nonnegative $g \in \mathcal{L}^\infty$ such that $x \in Z_g$ and for all measurable sets A with $\int_A g\, d\sigma < \sigma(A)$ we have $\int_A f\, d\sigma < \varepsilon \cdot \sigma(A)$. Let $S = \left\{g \leq \frac{1}{2}\right\}$. Then $x \in e(S)$. For each subset $A \subseteq S$, with $\sigma(A) > 0$, we have $\int_A g\, d\sigma < \frac{1}{2} \cdot \sigma(A)$, and thus $\int_A f d\sigma < \varepsilon \cdot \sigma(A)$. It follows that $\sigma(\{y \in S : f(y) > \varepsilon\}) = 0$. Therefore, x is in the image under e of the set

$$S \cap \complement \{y \in X : f(y) > \varepsilon\} = \{y \in S : f(y) \leq \varepsilon\} \subseteq \{y \in X : f(y) \leq \varepsilon\},$$

and so $x \in e(\{y \in X : f(y) \leq \varepsilon\})$. □

Corollary 8.5. *A function $f \in \mathcal{L}^\infty$ is essentially close to $f(x)$ for σ-a.e. $x \in X$.*

9. A Lifting of $\mathcal{L}^\infty$ Capturing Approximate Values

Definition 9.1. A **multiplicative lifting** of $\mathcal{L}^\infty$ is a linear mapping T from $\mathcal{L}^\infty$ into $\mathcal{L}^\infty$ such that for each $f,\, g \in \mathcal{L}^\infty$,

a) $T(f) = f$ σ-a.e.,

b) if $f = g$ σ-a.e., then $T(f) = T(g)$, and

c) $T(f \cdot g) = T(f) \cdot T(g)$.

Given a topology on X, we say the lifting is strong if we have $T(f)(x) = f(x)$ at each point x where f is continuous.

Remark 9.2. The multiplicativity property forces a lifting to be a positive map. We have used a local notion of a strong lifting, which forces globally continuous functions to be retained by T.

Remark 9.3. A lifting can be used to produce a positive zero-set mapping by setting $Z_{f\sigma}$ equal to the set where $T(f)$ equals zero. We have shown that a zero-set mapping can be used to construct a lower density. We will now give a simple construction using a lower density e to produce a lifting T of $\mathcal{L}^\infty$ such that T captures all approximate values. That is, $T(f)(x) = c$ if f is essentially close to c at x. The construction will again use a nonstandard enlargement of a structure containing X and the real numbers. We should note first that not every limit produced by a limit process can be captured by a multiplicative lifting.

Example 9.4. Let f be the characteristic function of the interval $[0, 1]$. Then the symmetric interval derivative of f with respect to Lebesgue measure at the point 0 is 1/2. If T is a multiplicative lifting of $\mathcal{L}^\infty$, we must have $(T(f))^2 = T(f^2) = T(f)$. We cannot, therefore, have $T(f)(0) = 1/2$.

Theorem 9.5. *Let e be a lower density on X, and let b be the associated upper density. Let ϕ be a base choice function determined by b and e. For each $f \in \mathcal{L}^\infty$ and each $x \in X$, set $T(f)(x) = t_\phi(f)(x) := \mathrm{st}\, {}^*f\,(\phi(x))$.*

1. *The mapping T is a multiplicative lifting of $\mathcal{L}^\infty$.*
2. *If f is essentially close to c at x, then $T(f)(x) = c$.*
3. *The lifting T is a strong lifting for the b-topology and any coarser topology.*

Proof. Statements 1 and 2 follow immediately from Proposition 6.11 and Corollary 8.5. Since an upper density is a strong base operator, by Theorem 6.14, the monad of any point x in the b-topology is $E_x \cup \{x\}$. Therefore, T is a strong lifting with respect to the b-topology and any topology coarser than the b-topology. □

Remark 9.6. Assume that the lower density e is obtained from measure differentiation with respect to balls. Then $\phi(x)$ is a point in the monad of x and also in the nonstandard extension of every set for which x is a point of density. In general, the mapping $x \mapsto \mathrm{st}\, {}^*f\,(\phi(x))$ gives a lifting for locally bounded measurable functions, and even for functions in $\mathcal{L}^1(\sigma)$ if one allows infinities and does not require multiplicativity at such points.

Remark 9.7. The authors have found a paper of Eifrig [8] in which he uses a rapid ultrafilter and a mapping $x \to B_x$, where B_x is an internal subset of our set E_x, to establishing a lifting for measurable sets. His mapping is essentially the same as the map we have constructed for the lifting of sets. However, while aproof of the existence of a rapid ultrafilter needs the Continuum Hypothesis or Martin's Axiom, our construction and proof of a lifting of functions in $\mathcal{L}^\infty$, retaining approximate values, uses just basic nonstandard analysis and the existence of an optimal differentiation basis.

Remark 9.8. (Added July 21, 1999.) For measures absolutely continuous with respect to a reference measure σ, we have shown that a zero-set mapping Z can be obtained in several steps from a lower density. For the existence of the latter, even in a noncomplete measure space, see [12]. For a class of measures M that contains measures singular with respect to σ, one can keep the zero sets small by adding the additional condition that for each $\nu \in M$, Z_ν is contained in a measurable set A_ν with $\nu(A_\nu) = 0$. (This follows automatically from our differentiation result if $\nu \ll \sigma$.) H. von Weizsäcker and the second author have shown that there exists such a zero-set mapping, where each zero-set Z_ν is itself measurable, when the underlying σ-algebra is countably generated. The proof is given in an appendix to [5].

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